Lynch, K. M., "The Rat as a Carrier of Dysenteric Amoeba." *Jour. Amer. Med. Assoc.*, **65**, 2232–2234 (1915).

Wenyon, C. M., "Observations of the Protozoa in the Intestine of Mice." *Arch. Prot.*, Suppl., **1**, pp. 169–201, pls. 10–12 (1907).

LINEAR CONNECTIONS OF A SPACE WHICH ARE DETERMINED BY SIMPLY TRANSITIVE CONTINUOUS GROUPS

By Luther Pfahler Eisenhart

Department of Mathematics, Princeton University

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1. The concept of parallelism introduced by Levi-Civita for general Riemannian manifolds of n dimensions has been extended by many writers to manifolds for which there is not an assigned metric. Weyl[1] used the term *affine connection* to define the relation between elements at different points of a space and used a linear connection in the definition of which the coefficients Γ^i_{jk} are symmetric in j and k. Equivalent definitions have been used by Eddington,[2] and by Veblen[3] and the author[3] in their papers on the *geometry of paths*. Schouten[4] has made an analysis of types of linear connection without the restriction that the coefficients be symmetric. It is the purpose of this note to show that the infinitesimal generators of a simply transitive continuous group in n variables serve to define an unsymmetric linear connection for which there exist symmetric tensors g_{ij}, involving $n(n + 1)/2$ arbitrary constants, whose first covariant derivatives are zero. Moreover, if one of these tensors is taken for the fundamental tensor of a Riemannian manifold, the group is a transitive group of motions for the manifold.

2. Consider a space of n dimensions of coördinates x^i for $i = 1, \ldots, n$. Let $\lambda^i_{\alpha/}$ be the components of n linearly independent contravariant vectors in the space; in this notation a $\lambda^i_{\alpha/}$ given α for $\alpha = 1, \ldots, n$ indicates the vector and i for $i = 1, \ldots, n$ the component. Since the vectors are linearly independent, their determinant $\Lambda = |\lambda^i_{\alpha/}|$ is different from zero. Let Λ^α_i be the cofactor of $\lambda^i_{\alpha/}$ in Λ divided by Λ; then

$$\lambda^i_{\alpha/}\,\Lambda^\beta_i = \epsilon^\beta_\alpha, \tag{2.1}$$

where i is summed from 1 to n, according to the customary convention which will be followed in this paper, and where ϵ^β_α is 1 for $\alpha = \beta$ and 0 for $\alpha \neq \beta$.

If we denote by $\lambda'^i_{\alpha/}$ the components of the vector $\lambda^i_{\alpha/}$ in a coördinate system x'^i, we have

$$\lambda^i_{\alpha/} = \lambda'^p_{\alpha/} \frac{\partial x^i}{\partial x'^p}. \tag{2.2}$$

From (2.1) and analogous equations for the x''s, we have

$$\lambda^i_{\alpha/} \Lambda^\beta_i = \lambda'^p_{\alpha/} \Lambda'^\beta_p,$$

which are reducible by (2.2) to

$$\lambda'^p_{\alpha/} \left(\Lambda^\beta_i \frac{\partial x^i}{\partial x'^p} - \Lambda'^\beta_p \right) = 0$$

Since determinant $\Lambda' \neq 0$, we have

$$\Lambda'^\beta_p = \Lambda^\beta_i \frac{\partial x^i}{\partial x'^p}. \tag{2.3}$$

Hence for each value of β, the quantities Λ^β_i are the components of a covariant vector, i indicating the component.

3. From (2.3) we have by differentiation

$$\frac{\partial \Lambda'^\beta_p}{\partial x'^q} = \frac{\partial \Lambda^\beta_i}{\partial x^j} \frac{\partial x^j}{\partial x'^q} \frac{\partial x^i}{\partial x'^p} + \Lambda^\beta_i \frac{\partial^2 x^i}{\partial x'^p \partial x'^q}.$$

Multiplying by

$$\lambda'^r_{\beta/} = \lambda^k_{\beta/} \frac{\partial x'^r}{\partial x^k},$$

and summing for β, we get

$$\lambda'^r_{\beta/} \frac{\partial \Lambda'^\beta_p}{\partial x'^q} = \lambda^k_{\beta/} \frac{\partial \Lambda^\beta_i}{\partial x^j} \frac{\partial x^i}{\partial x'^p} \frac{\partial x^j}{\partial x'^q} \frac{\partial x'^r}{\partial x^k} + \frac{\partial x'^r}{\partial x^k} \frac{\partial^2 x^k}{\partial x'^p \partial x'^q}. \tag{3.1}$$

If we put

$$\bar{\Gamma}^k_{ij} = \lambda^k_{\beta/} \frac{\partial \Lambda^\beta_i}{\partial x^j}, \tag{3.2}$$

equations (3.1) become

$$\bar{\Gamma}'^r_{pq} = \bar{\Gamma}^k_{ij} \frac{\partial x^i}{\partial x'^p} \frac{\partial x^j}{\partial x'^q} \frac{\partial x'^r}{\partial x^k} + \frac{\partial x'^r}{\partial x^k} \frac{\partial^2 x^k}{\partial x'^p \partial x'^q}. \tag{3.3}$$

These are the conditions which the functions $\bar{\Gamma}^k_{ij}$ must satisfy in order that they determine a linear connection, as Schouten[4] has shown. This particular result is due to Weitzenböck.[5]

From the definition of Λ^β_i we have also

$$\lambda^k_{\beta/} \Lambda^\beta_i = \delta^k_i \tag{3.4}$$

where δ^i_k is 0 or 1 according as $i \neq k$ or $i = k$. From this equation and (3.2), we have also

$$\bar{\Gamma}^k_{ij} = - \Lambda^\beta_i \frac{\partial \lambda^k_{\beta/}}{\partial x^j}. \tag{3.5}$$

Multiplying (3.2) and (3.5) by Λ^α_k and $\lambda^i_{\alpha/}$, respectively, and summing for k and i in these respective cases, we have in consequence of (2.1)

$$\frac{\partial \Lambda^\alpha_i}{\partial x^j} = \Lambda^\alpha_k \bar{\Gamma}^k_{ij}, \tag{3.6}$$

and

$$\frac{\partial \lambda^k_{\alpha/}}{\partial x^j} = - \lambda^i_{\alpha/} \bar{\Gamma}^k_{ij}. \tag{3.7}$$

Since these are the conditions that the first covariant derivative of Λ^α_i and $\lambda^k_{\alpha/}$ with respect to the $\bar{\Gamma}$'s be zero, we say that for the linear connection defined by (3.2) each field of vectors $\lambda^k_{\alpha/}$ (and Λ^α_i) forms a *parallel field*.

4. If in place of (3.2) and (3.5) we put

$$\Gamma^k_{ij} = \lambda^k_{\beta/} \frac{\partial \Lambda^\beta_j}{\partial x^i} = - \Lambda^\beta_j \cdot \frac{\partial \lambda^k_{\beta/}}{\partial x^i}, \tag{4.1}$$

the Γ's satisfy equations of the form (3.3) and consequently define another linear connection. In this case we have in place of (3.6) and (3.7)

$$\frac{\partial \Lambda^\alpha_j}{\partial x^i} = \Lambda^\alpha_k \Gamma^k_{ij} \tag{4.2}$$

and

$$\frac{\partial \lambda^k_{\alpha 1}}{\partial x^i} = - \lambda^j_{\alpha 1} \Gamma^k_{ij}. \tag{4.3}$$

For this linear connection the vectors $\lambda_{\alpha/j}$ and Λ^α_k do not form parallel fields.

In order that there may exist a symmetric tensor g_{ij} whose first covariant derivative with respect to the Γ's be zero, we must have

$$\frac{\partial g_{ij}}{\partial x^k} - g_{lj} \Gamma^l_{ik} - g_{il} \Gamma^l_{jk} = 0 \tag{4.4}$$

The conditions of integrability of these equations are

$$g_{ih} B^h_{jkl} + g_{hj} B^h_{ikl} = 0, \tag{4.5}$$

where

$$B^h_{jkl} = \frac{\partial \Gamma^h_{jl}}{\partial x^k} - \frac{\partial \Gamma^h_{jk}}{\partial x^l} + \Gamma^m_{jl} \Gamma^h_{mk} - \Gamma^m_{jk} \Gamma^h_{ml}. \tag{4.6}$$

In order that equations (4.4) be completely integrable, that is that they admit a solution involving $n(n + 1)/2$ arbitrary constants, it is necessary and sufficient that all the functions B^h_{jkl} be zero. We show in the next section that these conditions are satisfied, when the vectors $\lambda^i_{\alpha/}$ determine the infinitesimal generators of a simply transitive continuous group in n variables.

5. If

$$X_\alpha f \equiv \lambda^i_{\alpha/} \frac{\partial f}{\partial x^i} \tag{5.1}$$

are the generators of a simply transitive group, it is necessary that

$$(X_\alpha, X_\beta) f \equiv \lambda^i_{\alpha/} \frac{\partial}{\partial x^i}\left(\lambda^j_{\beta/} \frac{\partial f}{\partial x^j}\right) - \lambda^j_{\beta/} \frac{\partial}{\partial x^j}\left(\lambda^i_{\alpha/} \frac{\partial f}{\partial x^i}\right) = C^\gamma_{\alpha\beta} X_\gamma f, \tag{5.2}$$

where the C's are constants. If these constants satisfy the equations

$$C^\sigma_{\alpha\beta}\, C^\delta_{\sigma\lambda} + C^\sigma_{\lambda\alpha}\, C^\delta_{\sigma\beta} + C^\sigma_{\beta\lambda}\, C^\delta_{\sigma\alpha} = 0 \quad (\alpha,\beta,\lambda,\delta,\sigma = 1,..,n) \tag{5.3}$$

the conditions are sufficient as well as necessary.[6]

From (5.2) we have

$$\lambda^k_{\alpha/} \frac{\partial \lambda^h_{\beta/}}{\partial x^k} - \lambda^k_{\beta/} \frac{\lambda\partial^h_{\alpha/}}{\partial x^k} = C^\gamma_{\alpha\beta}\, \lambda^h_{\gamma/}.$$

Multiplying this equations by $\Lambda^\alpha_i\, \Lambda^\beta_j$ and summing for α and β, we have in consequence of (3.4) and (4.1)

$$\Gamma^h_{ij} - \Gamma^h_{ji} = -\, C^\gamma_{\alpha\beta}\, \lambda^h_{\gamma/}\, \Lambda^\alpha_i\, \Lambda^\beta_j. \tag{5.4}$$

By means of these equations, equations (4.6) can be put in the form

$$B^h_{jkl} = \frac{\partial \Gamma^h_{lj}}{\partial x^k} - \frac{\partial \Gamma^h_{kj}}{\partial x^l} - C^\gamma_{\alpha\beta}\left[\frac{\partial}{\partial x^k}(\lambda^h_{\gamma/}\, \Lambda^\alpha_j\, \Lambda^\beta_l) - \frac{\partial}{\partial x^l}(\lambda^h_{\gamma/}\, \Lambda^\alpha_j\, \Lambda^\beta_k)\right] + \Gamma^m_{jl}\, \Gamma^h_{mk} - \Gamma^m_{jk}\, \Gamma^h_{ml}.$$

Substituting in the first two terms expressions for the Γ's given by the second of (4.1), the resulting expressions are reducible to zero by means of (4.2), (4.3), (5.3) and (5.4).

6. If g_{ij} are the components of the fundamental tensor of a Riemannian manifold V_n of n dimensions, and $\lambda^i_{\alpha/}$ for $i = 1,..,n$ are a set of functions satisfying the equations

$$\lambda^l_{\alpha/} \frac{\partial g_{ij}}{\partial x^l} + g_{ih} \frac{\partial \lambda^h_{\alpha/}}{\partial x^j} + g_{hj} \frac{\partial \lambda^h_{\alpha/}}{\partial x^i} = 0, \tag{6.1}$$

then, as Killing[7] has shown, $X_\alpha f = \lambda^i_{\alpha/} \frac{\partial f}{\partial x^i}$ is the generator of a one parameter group of motions of V_n into itself. Conversely, suppose that (5.1) are the generators of a simply transitive group. In order that this be a group of motions for a V_n, it is necessary that (6.1) for $\alpha = 1,...,n$ hold. Multiplying (6.1) by Λ^α_k and summing for α, we get (4.4) in consequence of (3.4) and (4.1) Hence equations (6.1) admit solutions involving $n(n+1)/2$ arbitrary constants, and consequently there are $\infty^{\frac{n(n+1)}{2}}$ Riemannian manifolds admitting a given simply transitive group as a group of motions, a result due to Bianchi.[8]

[1] Space, Time, Matter, p. 112.

[2] The Mathematical Theory of Relativity, Chapter 7.

[3] These PROCEEDINGS, **8** and **9**; *Trans. Amer. Soc.*, **25** and **26**; *Annals*, **24.**

[4] Der Ricci-Kalkül, pp. 64, 65.

[5] Invariantentheorie, pp. 318, 319.

[6] Lie, Vorlesungen über Continuerliche Gruppen, pp. 391, 396.

[7] Crelle, **109**, 1892, p. 121.

[8] Lezioni Sulla Teoria dei Gruppi Continue Finiti, p. 517.

NOTE ON A THEOREM BY H. KNESER

BY J. W. ALEXANDER

DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY

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In a recent number of the *Proceedings of the Royal Academy of Amsterdam*,[1] H. Kneser proves the following theorem:

I. *A necessary and sufficient condition that a connected n-dimensional manifold M^n $(n > 1)$ be separated into two and only two regions by every connected $(n-1)$-dimensional manifold M^{n-1} contained in M^n is that the first Betti number P^1 of M^n be equal to unity and that there be no even coefficients of torsion of the lowest order.*

Kneser's proof is combinatorial and presupposes that M^{n-1} always belongs to a cellular subdivision of M^n.

It is perhaps worth noticing that the meaning of this theorem becomes very transparent in the light of the simplified, modulo 2 theory of connectivity, in which no distinction is made between positively and negatively oriented cells. Let us say that an n-complex is *completely connected* if it is possible to pass from any n-cell of the complex to any other by a series of steps at each of which we go from an n-cell E^n to an n-cell F^n incident to the same $(n-1)$-cell as E^n. Then, in the language of the modulo 2 theory, we have the following basic theorem which is almost self-evident.

II. *Let C^n be any closed, irreducible n-complex and C^{n-1} any closed irreducible $(n-1)$-complex made up of cells of C^n. Then, if C^{n-1} is bounding, it decomposes the complex C^n into exactly two completely connected complexes C_1^n and C_2^n, but if C^{n-1} is non-bounding, it leaves C^n completely connected.* The complexes C_1^n and C_2^n which touch along C^{n-1} may also touch in certain cells of dimensionalities less than $n-1$.